

Research Article

Midpoint-Free Subsets of the Real Numbers

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A set of reals $S \subset \mathbb{R}$ is *midpoint-free* if it has no subset $\{a, b, c\} \subseteq S$ such that $a < b < c$ and $a + c = 2b$. If $S \subset X \subseteq \mathbb{R}$ and S is midpoint-free, it is a *maximal midpoint-free subset* of X if there is no midpoint-free set T such that $S \subset T \subseteq X$. In each of the cases $X = \mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q}, \mathbb{R}^+, \mathbb{R}$, we determine two maximal midpoint-free subsets of X characterised by digit constraints on the base 3 representations of their members.

1. Introduction

Let us say that a set of three real numbers $\{a, b, c\} \subset \mathbb{R}$ is a *midpoint triple* if it satisfies $a < b < c$ and $a + c = 2b$. In this case b is the *midpoint* of the set, a is its *lower endpoint*, and c is its *upper endpoint*. This geometric viewpoint immediately suggests the main objective of this paper, which is to identify several significant subsets of $\mathbb{R}$ that contain no midpoint triple.

A *midpoint-free* subset of $\mathbb{R}$ is any set that contains no midpoint triple. For instance, the set $\{2^n : n \in \mathbb{Z}\}$ of powers of 2 is midpoint-free, since appropriate scaling shows that the sum of two distinct powers of 2 is never equal to a power of 2. Note that if $S \subseteq \mathbb{R}$ and $cS + d := \{cs + d : s \in S\}$, then $cS + d$ is midpoint-free, for any $\{c, d\} \subset \mathbb{R}$ with $c \neq 0$, if and only if S is midpoint-free.

If $S \subset X \subseteq \mathbb{R}$ and S is midpoint-free, then S is a *maximal midpoint-free subset* of X if there is a midpoint triple in any set T such that $S \subset T \subseteq X$. (In this case, any $x \in X \setminus S$ is a member of a midpoint triple in $S \cup \{x\}$.) If X is a “natural” subset of $\mathbb{R}$, such as the nonnegative integers $\mathbb{Z}^+$, the rationals $\mathbb{Q}$, or indeed $\mathbb{R}$ itself, it is of considerable interest to identify maximal midpoint-free subsets of X .

Alternatively, any set $\{a, b, c\}$ such that $a < b < c$ and $a + c = 2b$ can be viewed as a 3-term arithmetic progression (A.P.) with first term a , midterm b , and last term c . This arithmetic viewpoint has led to many studies of sequences of positive integers in which there is no 3-term A.P., raising questions such as the following. How large is a

maximal subset of the first n positive integers that contains no 3-term A.P.? (See [1] for data.) Given a finite set S of positive integers with no 3-term A.P., what does its greedy algorithm extension look like? How does it compare with a largest possible subset of the first n positive integers that contains S and has no 3-term A.P.? Under what conditions must a subset of the positive integers contain a 3-term A.P.? Among those who have made major contributions to our understanding of these questions, one must list such luminaries as Van der Waerden, Erdős, Turán, Rado, Behrend, Roth, Graham, and Szemerédi. For a compact survey and extensive bibliography of such investigations, see Guy [2]. As an example of recent work in this area, see Dybizbański [3].

In contrast with the usual arithmetic viewpoint, the geometric viewpoint adopted here takes us in an apparently novel direction, where the sets of interest are most naturally viewed as subsets of $\mathbb{R}$.

2. Notational Conventions

In what follows, a key tool for discussing midpoint triples will be base 3 representations of the real numbers, so relevant notational conventions will now be specified. If $r \in \mathbb{R}$, $r > 0$ and

$$r = \sum_{i \in \mathbb{Z}} d_i 3^i \quad \text{with } d_i \in \{0, 1, 2\}, \quad (1)$$

then the two-way infinite string

$$\mathbf{d} := \cdots d_2 d_1 d_0 \cdot d_{-1} d_{-2} \cdots \quad (2)$$

is a base 3 representation of r and d_i is the *digit in place* i of $\mathbf{d}$. If $d_i \in \{0, 1\}$ for infinitely many $i < 0$, then $\mathbf{d}$ is a *regular* base 3 representation of r ; otherwise, $\mathbf{d}$ is a *singular* base 3 representation of r , and there is an integer k such that $d_i = 2$ for all $i < k$. The regular base 3 representation of r is unique, and if r has a singular base 3 representation, that is also unique. If X is some “natural” subset of $\mathbb{R}$ and $D \subset \{0, 1, 2\}$, it will be convenient to use $X_3(D)$ to denote the set of all members of X with a base 3 representation restricted to D . This notation adapts to cases where D is a configuration of base 3 digits; thus, when D is a finite string of base 3 digits, $X_3([D])$ will denote the set of all members of X with a base 3 representation which includes a one-way infinite string of recurring blocks D .

Since $r > 0$, at least one of the digits is nonzero. The *leading digit* of $\mathbf{d}$ is the digit $d_h > 0$, such that $d_i = 0$ for all $i > h$. The *trivial zeros* of $\mathbf{d}$ are all the digits $d_i = 0$ with $i > \max\{h, 0\}$. Conventionally these are suppressed (i.e., kept implicit) when listing $\mathbf{d}$. If $h < 0$, so the leading digit d_h occupies a negative place, the *placeholder zeros* of $\mathbf{d}$ are all the digits $d_i = 0$ with $0 \geq i > h$. If there is an integer $k \leq h$ such that $d_k > 0$ and $d_i = 0$ for all $i < k$, then $\mathbf{d}$ is a regular representation, d_k is its *trailing digit*, its *optional zeros* are all the digits $d_i = 0$ with $i < \min\{k, 0\}$, and if $k > 0$ then its *placeholder zeros* are the digits $d_i = 0$ with $0 \leq i < k$. If $\mathbf{d}$ has a trailing digit, it is conventional to suppress its optional zeros as well as its trivial zeros; the finite digit string remaining is the *terminating* base 3 representation of r .

These conventions are extended to $r = 0$, as follows. Although the digits in this case are $d_i = 0$ for all i , in this exceptional case d_0 is defined to be both the *leading digit* and the *trailing digit* of the representation. Then its trivial zeros are all those in places $i > 0$, and its optional zeros are all those in places $i < 0$; thus, 0 is the terminating representation for $r = 0$.

If $\mathbf{d}$ has no trailing digit, it is a *nonterminating* representation of r and it may be either regular or singular. If $\mathbf{d}$ is singular, there is an integer $k \leq h + 1$ such that $d_k \in \{0, 1\}$ and $d_i = 2$ for all $i < k$; the digit d_k is the *pivot digit* of $\mathbf{d}$. In this case there is an alternative, regular base 3 representation of r , namely,

$$\mathbf{e} := \cdots e_2 e_1 e_0 \cdot e_{-1} e_{-2} \cdots, \quad (3)$$

where $e_i = d_i$ for all $i > k$, $e_k = d_k + 1$, and $e_i = 0$ for all $i < k$. Evidently $\mathbf{e}$ is a terminating representation, e_k is its trailing digit, and $3^{-k}r \in \mathbb{Z}^+$, so $r \in 3^k \mathbb{Z}^+$.

The regular base 3 representation of any nonnegative rational $q \in \mathbb{Q}^+$ has a recurring block of digits; if q has a singular base 3 representation, this has a recurring 2. For brevity, any such recurring block can be enclosed in square

brackets, and the subscript 3 can be used to indicate base 3 notation; for instance,

$$\begin{aligned} \frac{7}{3} &= 2 \cdot 1_3 = 2 \cdot 1[0]_3 = 2 \cdot 0[2]_3, \\ \frac{1}{2} &= 0 \cdot [1]_3, \quad \frac{2}{5} = 0 \cdot [1012]_3. \end{aligned} \quad (4)$$

The set of all nonnegative rationals with a terminating base 3 representation is

$$\mathbb{Q}_3^+([0]) = \bigcup_{i \in \mathbb{Z}^+} 3^{-i} \mathbb{Z}^+. \quad (5)$$

The set of positive rationals with a singular base 3 representation is simply

$$\mathbb{Q}_3^+([2]) = \mathbb{Q}_3^+([0]) \setminus \{0\}. \quad (6)$$

The set of positive rationals with base 3 representation containing a recurring 1 is

$$\mathbb{Q}_3^+([1]) = \bigcup_{i \in \mathbb{Z}^+} 3^{-i} \left(\mathbb{Z}^+ + \frac{1}{2} \right). \quad (7)$$

It will later be found that this set requires particular attention when members are doubled, because

$$2\mathbb{Q}_3^+([1]) \subset \mathbb{Q}_3^+([2]) \subset \mathbb{Q}_3^+([0]). \quad (8)$$

Base 3 representation of negative reals is simply achieved by writing any such number as $-r$, with $r \in \mathbb{R}$, $r > 0$, so that $-r$ has its base 3 representation adopted from that of r , with trivial zeros suppressed and the unary minus operator applied to precede the leading digit. If r has both a regular and a singular representation, so does $-r$.

To distinguish between instances of base 3 and base 10 representations, an explicit subscript 3 is normally attached to the former, while a subscript 10 is normally kept implicit for the latter. As needed, r_3 will denote the regular base 3 representation of r , with r_3^* reserved for the singular representation when this exists. Finally, $\llbracket r \rrbracket_{3,i}$ will denote d_i , the digit in place i of the regular base 3 representation of r , while $\llbracket r \rrbracket_{3,i}^*$ will denote the corresponding digit in the singular representation when this exists.

3. Three Sparse Subsets of $\mathbb{Z}^+$

Let $\mathbb{Z}_3^+(D)$ be the set of all nonnegative integers with terminating base 3 representation in which the only explicit digits are in $D \subset \{0, 1, 2\}$. The three cases of interest are those where D is a 2-set. They begin as follows:

$$\mathbb{Z}_3^+(0, 1)$$

$$= \{0, 1, 3, 4, 9, 10, 12, 13, 27, 28,$$

$$30, 31, 36, 37, 39, 40, 81, \dots\},$$

$$\mathbb{Z}_3^+(0, 2)$$

$$= \{0, 2, 6, 8, 18, 20, 24, 26, 54, 56, 60, 62, 72, 74, 78, 80, \dots\},$$

$$\begin{aligned} \mathbb{Z}_3^+(1, 2) \\ = \{1, 2, 4, 5, 7, 8, 13, 14, 16, 17, 22, 23, 25, 26, 40, \dots\}. \end{aligned} \quad (9)$$

Some features are obvious. Clearly $\mathbb{Z}_3^+(0, 2) = 2\mathbb{Z}_3^+(0, 1)$. When $d \in \{0, 1, 2\}$ and $D = \{0, 1, 2\} \setminus \{d\}$, all integers congruent to $d \pmod{3}$ are absent from $\mathbb{Z}_3^+(D)$. The first positive integer absent from all three sets is $102_3 = 11$, but it is clear that an increasing proportion of integers will be missing from all three sets. Indeed, a simple calculation shows for any positive integer n that $\mathbb{Z}_3^+(0, 1)$ and $\mathbb{Z}_3^+(0, 2)$ each have 2^n members below 3^n , while $\mathbb{Z}_3^+(1, 2)$ has $2^{n+1} - 2$ members below 3^n , so each set is sparse and has asymptotic density 0.

It turns out that these three sets are of considerable interest when it comes to the presence or absence of midpoint triples, so let us now address the main subject of this paper.

Note that the set $\mathbb{Z}_3^+(1, 2)$ contains midpoint triples, such as $\{1, 4, 7\}$ and $\{2, 5, 8\}$. In fact, $\mathbb{Z}_3^+(1, 2)$ is densely packed with midpoint triples, since each member $a \in \mathbb{Z}_3^+(1, 2)$ is the lower endpoint of an infinite family of midpoint triples:

$$\begin{aligned} \{a, u_n - u_m + a, 2u_n - 2u_m + a\} \\ = (u_n - u_m) \{0, 1, 2\} + a \subset \mathbb{Z}_3^+(1, 2) \end{aligned} \quad (10)$$

for any integer $n > m$, where m is the number of digits in the terminating base 3 representation of a (if d_h is the leading digit in a_3 , then $m = h + 1$), and

$$u_n := 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2} \quad (11)$$

is the positive integer with terminating base 3 representation of n digits, all 1. Note that $\mathbb{Z}_3^+(1) = \{u_n : n \geq 1\}$, and appropriate scaling shows that this set is midpoint-free. (The integers u_n are *rep-units* in base 3. The positive integers with explicit decimal digits all equal to 1 are the *rep-units* for base 10, so called by contraction of “repeated unit” [4]. Prime factorizations of rep-units have been much studied [5, 6]. Clearly, if $m \mid n$, then $u_m \mid u_n$. Thus u_n can be a prime number only if n is prime, but $u_5 = 11111_3 = 11^2$ shows that this condition is not sufficient. Such observations generalize to rep-units in any base.)

Not only is every member of $\mathbb{Z}_3^+(1, 2)$ the lower endpoint of infinitely many midpoint triples, it is also the case that any member greater than 2 is either the midpoint or the upper endpoint of at least one midpoint triple, depending on the leading digit of its base 3 representation.

In contrast, it turns out that $\mathbb{Z}_3^+(0, 1)$ and $\mathbb{Z}_3^+(0, 2)$ are midpoint-free. To see this, it suffices to show that $\mathbb{Z}_3^+(0, 1)$ is midpoint-free: then $\mathbb{Z}_3^+(0, 2) = 2\mathbb{Z}_3^+(0, 1)$ ensures that $\mathbb{Z}_3^+(0, 2)$ is also midpoint-free. (Long ago, Erdős and Turán [7] noted that the nonnegative integers with a 2-less base 3 representation are midpoint-free. The following compact proof is included for completeness and to typify what follows.)

Claim A. The set $\mathbb{Z}_3^+(0, 1)$ is midpoint-free.

Proof. On the contrary, suppose that $\{a, b, c\} \subset \mathbb{Z}_3^+(0, 1)$ is a midpoint triple, with $a < b < c$ and $a + c = 2b$. All digits in a_3, b_3, c_3 are in $\{0, 1\}$ and all digits in $(2b)_3$ are in $\{0, 2\}$. Then

$$[a]_{3,i} + [c]_{3,i} = [2b]_{3,i}, \quad (12)$$

for all integers i , and

$$\begin{aligned} [2b]_{3,i} = 0 &\implies [a]_{3,i} = [b]_{3,i} = [c]_{3,i} = 0, \\ [2b]_{3,i} = 2 &\implies [a]_{3,i} = [b]_{3,i} = [c]_{3,i} = 1. \end{aligned} \quad (13)$$

Hence, $a = b = c$, a contradiction, so $\mathbb{Z}_3^+(0, 1)$ contains no midpoint triple. $\square$

It will now be shown that $\mathbb{Z}_3^+(0, 1)$ is not contained in any larger midpoint-free subset of $\mathbb{Z}^+$.

Claim B. For any positive integer $b \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0, 1)$, there is a midpoint triple in $\mathbb{Z}_3^+(0, 1) \cup \{b\}$ with b as its midpoint.

Proof. Given $b \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0, 1)$, we seek $\{a, c\} \subset \mathbb{Z}_3^+(0, 1)$ such that $a < b < c$ and $a + c = 2b$. Specify the base 3 digits of a and c as follows:

$$\begin{aligned} [2b]_{3,i} = 0 &\implies [a]_{3,i} = [c]_{3,i} = 0, \\ [2b]_{3,i} = 2 &\implies [a]_{3,i} = [c]_{3,i} = 1, \\ [2b]_{3,i} = 1 &\implies [a]_{3,i} = 0, \quad [c]_{3,i} = 1. \end{aligned} \quad (14)$$

This determines integers a and c such that $\{a, c\} \subset \mathbb{Z}_3^+(0, 1)$ and $a + c = 2b$ since

$$[a]_{3,i} + [c]_{3,i} = [2b]_{3,i}, \quad (15)$$

for all i . At least one digit of $(2b)_3$ is 1, since $b \notin \mathbb{Z}_3^+(0, 1)$, so a_3 and c_3 differ in at least one place: then $[a]_{3,i} < [c]_{3,i}$ in each such place, ensuring that $a < c$. Put $d := (c - a)/2 > 0$. Then $c = a + 2d$, so $2b = a + c = 2a + 2d$, whence $b = a + d > a$ and $c = a + 2d = b + d > b$. Thus $a < b < c$. $\square$

It will now be shown that extending $\mathbb{Z}_3^+(0, 1)$ by adjoining any rational of the form $n + (1/2)$, with $n \in \mathbb{Z}^+$, always yields a set which contains a midpoint triple.

Claim C. For any $b \in \mathbb{Z}^+ + (1/2)$, there is a midpoint triple in $\mathbb{Z}_3^+(0, 1) \cup \{b\}$ with b as its midpoint.

Proof. Again we seek $\{a, c\} \subset \mathbb{Z}_3^+(0, 1)$, such that $a < b < c$ and $a + c = 2b$. In the present case, note that $2b$ is an odd positive integer, so $(2b)_3$ contains the digit 1 an odd number of times. Specifying a and c as in the proof of Claim B, once again it follows that $\{a, c\} \subset \mathbb{Z}_3^+(0, 1)$ and $a < b < c$. $\square$

For instance, the proofs of Claims B and C produce the decompositions

$$1201_3 = 100_3 + 1101_3, \quad 212_3 = 101_3 + 111_3 \quad (16)$$

showing that 23 and $23/2$ are the midpoints of the pairs $\{9, 37\} \subset \mathbb{Z}_3^+(0, 1)$ and $\{10, 13\} \subset \mathbb{Z}_3^+(0, 1)$, respectively.

Further, doubling shows that 46 and 23 are the midpoints of the pairs $\{18, 74\} \subset \mathbb{Z}_3^+(0, 2)$ and $\{20, 26\} \subset \mathbb{Z}_3^+(0, 2)$, respectively. From Claims A, B, and C and the doubling method just illustrated, we have the following.

Theorem 1. *The sets $\mathbb{Z}_3^+(0, 1)$ and $\mathbb{Z}_3^+(0, 2)$ are maximal midpoint-free subsets of $\mathbb{Z}^+$.*

4. Two Unexpected Results for $\mathbb{Z}$

Let us now study $\mathbb{Z}_3^+(0, 1)$ and $\mathbb{Z}_3^+(0, 2)$ as midpoint-free subsets of $\mathbb{Z}$, the full set of integers. Surprisingly, the results for the two sets are quite different.

Calculations until now have involved sums of pairs $\{a, c\} \subset \mathbb{Z}_3^+(0, 1)$, so it has been possible to work digit by digit in base 3 without a carry over digit. When the context is widened to $\mathbb{Z}$, it seems that carry over digits can no longer be avoided, so we shall consider blocks of digits rather than single digits. Let the compact notation $d^{\times n}$ (“ d by n ”) denote a homogeneous block of n adjacent digits all equal to d ; thus, $1^{\times n}$ denotes the terminating base 3 representation of rep-unit u_n . For blocks in which the digits are not all equal we simply abut such expressions with $d^{\times 1} = d$ as a natural simplification. To illustrate, consider the integer

$$a_3 := 1222202111001121100 = 12^{\times 4} 021^{\times 3} 0^{\times 2} 1^{\times 2} 21^{\times 2} 0^{\times 2}. \quad (17)$$

Let us find integers $b, c \in \mathbb{Z}_3^+(0, 1)$ such that $a + 2b = c$. Regarding a_3 as a string of homogeneous blocks, we choose matching blocks for $(2b)_3$ so that

$$(2b)_3 := 200002022222202200 = 20^{\times 4} 202^{\times 7} 02^{\times 2} 0^{\times 2}. \quad (18)$$

Calculating from right to left, base 3 arithmetic for $a_3 + (2b)_3$ yields successively the following blocks, with each carry over digit indicated parenthetically:

$$\begin{aligned} 0^{\times 2} + 0^{\times 2} &= 0^{\times 2}, \\ 1^{\times 2} + 2^{\times 2} &= (1) 10, \\ 2 + 0 + (1) &= (1) 0, \\ 1^{\times 2} + 2^{\times 2} + (1) &= (1) 1^{\times 2}, \\ 0^{\times 2} + 2^{\times 2} + (1) &= (1) 0^{\times 2}, \\ 1^{\times 3} + 2^{\times 3} + (1) &= (1) 1^{\times 3}, \\ 2 + 0 + (1) &= (1) 0, \\ 0 + 2 + (1) &= (1) 0, \\ 2^{\times 4} + 0^{\times 4} + (1) &= (1) 0^{\times 4}, \\ 1 + 2 + (1) &= (1) 1. \end{aligned} \quad (19)$$

Assembling these blocks yields

$$a_3 + (2b)_3 = 1^{\times 2} \mid 0^{\times 4} \mid 0 \mid 0 \mid 1^{\times 3} \mid 0^{\times 2} \mid 1^{\times 2} \mid 0 \mid 10 \mid 0^{\times 2}. \quad (20)$$

After simplifying we have

$$10^{\times 4} 101^{\times 7} 01^{\times 2} 0^{\times 2} = 100001011111101100 = b_3,$$

$$1^{\times 2} 0^{\times 6} 1^{\times 3} 0^{\times 2} 1^{\times 2} 010^{\times 3} = 11000000111001101000 = c_3, \quad (21)$$

so $a + 2b = c$ and $b, c \in \mathbb{Z}_3^+(0, 1)$ as desired. This calculation models the proof of the following.

Claim D. For any integer $a > 0$, there is a midpoint triple in $\mathbb{Z}_3^+(0, 1) \cup \{-a\}$ with the negative integer $-a$ as its lower endpoint.

Proof. Given any positive integer a , we seek $\{b, c\} \subset \mathbb{Z}_3^+(0, 1)$, such that $-a < b < c$ and $-a + c = 2b$, so $a + 2b = c$. Partition a_3 into its homogeneous blocks A_i , so

$$a_3 = A_m \cdots A_i \cdots A_1 A_0 \quad (22)$$

and seek corresponding blocks B_i and C_i to construct b_3 and c_3 .

If a_3 has a terminal block $A_0 = 0^{\times n}$, assign b_3 the terminal block $B_0 = 0^{\times n}$, so c_3 has the terminal block $C_0 = 0^{\times n} = A_0 + 2B_0$. If a_3 has a nonterminal block $A_i = 0^{\times n}$ for some $i > 0$, assume the sum of the preceding blocks A_{i-1} and $2B_{i-1}$ contributes a carry over digit of 1, and assign b_3 the corresponding block $B_i = 1^{\times n}$, so

$$C_i = A_i + 2B_i + (1) = 0^{\times n} + 2^{\times n} + (1) = (1) 0^{\times n}. \quad (23)$$

As $2^{\times n} + (1)$ corresponds to $2u_n + 1 = (3^{n+1} - 1) + 1 = 3^{n+1}$, the computation is justified.

If a_3 has a block $A_i = 1^{\times n}$, assign b_3 the corresponding block $B_i = 1^{\times n}$, so c_3 has the corresponding block $C_i = A_i + 2B_i + (\delta) = (1) 1^{\times(n-1)} 0 + (\delta)$, where the carry over digit δ is either 0 or 1, so $C_i = (1) 1^{\times(n-1)} 0$ or $(1) 1^{\times n}$, respectively. This holds since the sum $1^{\times n} + 2^{\times n}$ corresponds to $u_n + 2u_n = 3u_n = u_{n+1} - u_1$ and $1^{\times n} + 2^{\times n} + (1)$ corresponds to $3u_n + 1 = u_{n+1}$.

If a_3 has a terminal block $A_0 = 2^{\times n}$, assign b_3 the block $B_0 = 0^{\times(n-1)} 1$, so c_3 has terminal block $C_0 = A_0 + 2B_0 = (1) 0^{\times(n-1)} 1$, since the sum $2^{\times n} + 0^{\times(n-1)} 2$ corresponds to $2u_n + 2 = (3^{n+1} - 1) + 2 = 3^{n+1} + 1$. Similarly if $A_1 A_0 = 2^{\times n} 0^{\times m}$, assign b_3 the corresponding blocks $B_1 B_0 = 0^{\times(n-1)} 10^{\times m}$; then c_3 has the blocks $C_1 C_0 = (1) 0^{\times(n-1)} 10^{\times m}$. Otherwise, if a_3 has a nonterminal block $A_i = 2^{\times n}$ for some $i > 0$, assume that $A_{i-1} + 2B_{i-1}$ yields a carry over digit 1, and assign b_3 the block $B_i = 0^{\times n}$, so c_3 has $C_i = A_i + 2B_i + (1) = (1) 0^{\times n}$.

For $i > 0$ note that the only case in which $A_{i-1} + 2B_{i-1}$ does not contribute a carry over digit 1 to $A_i + 2B_i$ is when $i = 1$ and a_3 has terminal block $A_0 = 0^{\times n}$. Thus, all our assumptions about carry over digits are justified *a posteriori*. $\square$

Claim E. For any integer $n > 0$, there is a midpoint triple in $\mathbb{Z}_3^+(0, 2) \cup \{-n\}$ if and only if n is even, and then $-n$ is the lower endpoint of the triple.

Proof. By doubling, it follows from Claim D that, for any integer $a > 0$, there is a midpoint triple in $\mathbb{Z}_3^+(0, 2) \cup$

$\{-2a\}$. However, we claim that there is no midpoint triple in $\mathbb{Z}_3^+(0, 2) \cup \{-2a + 1\}$. On the contrary, suppose $-2a + 1$ forms a midpoint triple with the pair $\{b, c\} \subset \mathbb{Z}_3^+(0, 2)$. Without loss of generality, $-2a + 1 < b < c$ and $(-2a + 1) + c = 2b$, so $2a + 2b - 1 = c$: this is impossible, since $2a + 2b - 1$ is odd and c is even. It follows that $\mathbb{Z}_3^+(0, 2) \cup \{-2a + 1\}$ is midpoint-free. $\square$

Claim F. The set $\mathbb{Z}_3^+(0, 2) \cup -(\mathbb{Z}_3^+(0, 2) + 1)$ is midpoint-free.

Proof. The two sets forming this union are certainly midpoint-free, so any midpoint triple in the union must have two members in one set and one member in the other. Claim E shows that there is no midpoint triple with exactly one member in $-(\mathbb{Z}_3^+(0, 2) + 1)$, since this set only contains odd negative integers. So suppose there is a midpoint triple $\{-a, -b, c\}$ with exactly one member $c \in \mathbb{Z}_3^+(0, 2)$. Then $\{a - 1, b - 1, -c - 1\}$ is a midpoint triple in

$$\begin{aligned} & -(\mathbb{Z}_3^+(0, 2) \cup -(\mathbb{Z}_3^+(0, 2) + 1)) - 1 \\ & = \mathbb{Z}_3^+(0, 2) \cup -(\mathbb{Z}_3^+(0, 2) + 1) \end{aligned} \quad (24)$$

with exactly one member in $-(\mathbb{Z}_3^+(0, 2) + 1)$, the case ruled out by Claim E. $\square$

Since $\mathbb{Z}_3^+(0, 2)$ is a maximal midpoint-free subset of $\mathbb{Z}^+$ by Theorem 1, clearly $\mathbb{Z}_3^+(0, 2) + 1$ is a maximal midpoint-free subset of $\mathbb{Z}^+ + 1 = \mathbb{Z}^+ \setminus \{0\}$. Note that the members of $\mathbb{Z}_3^+(0, 2) + 1$ are precisely those positive integers n for which the trailing digit of n_3 is 1, and all other digits are in $\{0, 2\}$. With Claims E and F, it follows that $\mathbb{Z}_3^+(0, 2) \cup -(\mathbb{Z}_3^+(0, 2) + 1)$ is a maximal midpoint-free subset of $\mathbb{Z} = \mathbb{Z}^+ \cup -(\mathbb{Z}^+ \setminus \{0\})$. Combined with Claim D, this proves the following.

Theorem 2. The sets $\mathbb{Z}_3^+(0, 1)$ and $\mathbb{Z}_3^+(0, 2) \cup -(\mathbb{Z}_3^+(0, 2) + 1)$ are maximal midpoint-free subsets of $\mathbb{Z}$.

5. Midpoint-Free Subsets of $\mathbb{Q}^+$

Now let us consider the corresponding subsets $\mathbb{Q}_3^+(D)$ of the nonnegative rationals $\mathbb{Q}^+$, namely, the subsets comprising those members with a regular base 3 representation in which each digit, after suppression of all trivial and optional zeros, is in $D \subset \{0, 1, 2\}$. The three cases of interest are those where D is a 2-set.

As expected, $\mathbb{Q}_3^+(1, 2)$ is densely packed with midpoint triples. This follows from the observation that each rational $q \in \mathbb{Q}_3^+(1, 2)$ is the lower endpoint of an infinite family of midpoint triples,

$$\begin{aligned} & \{q, u_n - u_m + q, 2u_n - 2u_m + q\} \\ & = (u_n - u_m) \{0, 1, 2\} + q \subset \mathbb{Q}_3^+(1, 2) \end{aligned} \quad (25)$$

for any integer $n > m$, where u_n is the n -digit base 3 rep-unit and $m = h + 1$, where d_h is the leading digit of the regular base 3 representation q_3 .

Next, consider $\mathbb{Q}_3^+(0, 1)$. The details are a little more complicated than in the integer context, since allowance

must be made for singular representations when particular members of $\mathbb{Q}_3^+(0, 1)$ are doubled.

Claim G. The set $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ is midpoint-free.

Proof. If $b \in \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ and $b > 0$, all digits of $(2b)_3$ are in $\{0, 2\}$. They uniquely determine the digits of a_3 and c_3 such that $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ and $a + c = 2b$. As in the proof of Claim A, they force $a = b = c$. Hence, there is no midpoint triple $\{a, b, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$. $\square$

Claim H. Any $b \in \mathbb{Q}_3^+([1])$ is the midpoint of a triple with both its endpoints in the set $\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([0])$.

Proof. We seek $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ with $a < b < c$ and $a + c = 2b$. Then a_3 and c_3 are uniquely determined by

$$[a]_{3,i} + [c]_{3,i} = [2b]_{3,i}, \quad [a]_{3,i} \leq [c]_{3,i} \quad (26)$$

for all i . Since $b \in \mathbb{Q}_3^+([1])$, then $3^k b \in \mathbb{Z}^+ + (1/2)$ for some $k \in \mathbb{Z}^+$, so

$$2b \in 3^{-k} (2\mathbb{Z}^+ + 1). \quad (27)$$

It follows that $(2b)_3$ is a terminating representation with an odd number of digits equal to 1. Then

$$\begin{aligned} [2b]_{3,i} = 1 & \implies [a]_{3,i} = 0, \quad [c]_{3,i} = 1 \\ & \text{for an odd number of integers } i \geq -k; \end{aligned} \quad (28)$$

$$[2b]_{3,i} = 0 \implies [a]_{3,i} = [c]_{3,i} = 0 \quad \forall i < -k.$$

Hence, the required midpoint triple $\{a, b, c\}$ exists, and in fact $\{a, c\} \subset \mathbb{Q}_3^+([0])$. $\square$

In particular, if $b \in \mathbb{Q}_3^+([1])$ is such that $[2b]_{3,n} = 1$ for some integer n and $[2b]_{3,i} = 0$ for all $i \neq n$, then $b = 3^n/2 = u_n + (1/2)$ and the proof of Claim H determines $a = 0$ and $c = 2b = 3^n$. Since $[b]_{3,i} = 0$ if $i \geq n$ and $[b]_{3,i} = 1$ if $i < n$, we call b the base 3 *fractional rep-unit with offset n* , denoted by v_n . Then $v_n \in \mathbb{Q}_3^+([1])$ is the midpoint of the triple with endpoints $\{0, 3^n\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([0])$.

Positive rationals not in $\mathbb{Q}_3^+(0, 1) \cup \mathbb{Q}_3^+([1])$ also belong to midpoint triples with endpoints in $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$, but proving this needs care. For instance,

$$\begin{aligned} \frac{5}{16} &= 0 \cdot [0221]_3 \\ &\implies \frac{5}{8} = 0 \cdot [12]_3 = 0 \cdot [0111]_3 + 0 \cdot [1101]_3 \\ &= \frac{13}{80} + \frac{37}{80}, \end{aligned} \quad (29)$$

so $\{13/80, 5/16, 37/80\}$ is a midpoint triple with endpoints in $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$. This calculation models the proof of the following general result.

Claim I. Any $b \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 1)$ is the midpoint of a triple with both its endpoints in the set $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$.

Proof. We seek $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ with $a < b < c$ and $a + c = 2b$. When $b \in \mathbb{Q}_3^+([1])$, the result follows from Claim H, so we may now assume that

$$b \in \mathbb{Q}^+ \setminus (\mathbb{Q}_3^+(0, 1) \cup \mathbb{Q}_3^+([1])). \quad (30)$$

Then $(2b)_3$ has at least one digit d_j equal to 1 and its recurring block, say $[D]$, has at least one digit different from 2. We require a_3 and c_3 to satisfy

$$\llbracket a \rrbracket_{3,i} + \llbracket c \rrbracket_{3,i} = \llbracket 2b \rrbracket_{3,i} \quad (31)$$

for all i . Requiring $\llbracket a \rrbracket_{3,i} \leq \llbracket c \rrbracket_{3,i}$ for all $i \geq j$ ensures that $a < c$, since

$$\llbracket a \rrbracket_{3,j} = 0, \quad \llbracket c \rrbracket_{3,j} = 1. \quad (32)$$

If the recurring block $[D]$ of $(2b)_3$ contains 0, then the recurring blocks of a_3 and c_3 also contain 0, so neither a nor c is in $\mathbb{Q}_3^+([1])$. Now suppose $[D]$ has at least one digit equal to 1 and none equal to 0. Clearly, in this case we may choose the integer j so that the digit $d_j = 1$ occurs in $[D]$. If D has m digits, then for all $i \leq j$ choose

$$\llbracket a \rrbracket_{3,i} = 0, \quad \llbracket c \rrbracket_{3,i} = 1 \quad \text{when } i \equiv j \pmod{2m},$$

$$\llbracket a \rrbracket_{3,i} = 1, \quad \llbracket c \rrbracket_{3,i} = 0 \quad \text{when } i \equiv j + m \pmod{2m}, \quad (33)$$

$$\llbracket a \rrbracket_{3,i} \leq \llbracket c \rrbracket_{3,i} \quad \text{otherwise.}$$

Then a_3 and c_3 are uniquely determined, so that $a < c$ and each has a recurring block containing 0, so neither a nor c is in $\mathbb{Q}_3^+([1])$. $\square$

If $q \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2)$ then q_3 has at least one digit equal to 1, so it follows that $(q/2)_3 \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 1)$. Claim I shows that there is a midpoint triple $\{a, b, c\}$ with $b = q/2$ and $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$. Then $2\{a, b, c\}$ is a midpoint triple with midpoint q and endpoints $2\{a, c\} \subset \mathbb{Q}_3^+(0, 2)$. Hence, doubling applied to Claims G and I shows that $\mathbb{Q}_3^+(0, 2)$ is midpoint-free, and every $b \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2)$ is the midpoint of a triple with endpoints in $\mathbb{Q}_3^+(0, 2)$, so Claims G, H, and I establish the following.

Theorem 3. *The sets $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ and $\mathbb{Q}_3^+(0, 2)$ are maximal midpoint-free subsets of $\mathbb{Q}^+$.*

6. Midpoint-Free Subsets of $\mathbb{Q}$

Now consider $\mathbb{Q}_3^+(0, 1)$ and $\mathbb{Q}_3^+(0, 2)$ as subsets of $\mathbb{Q}$. The role of $\mathbb{Q}_3^+([1])$ is yet more prominent in this context.

Claim J. The set $\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$ is midpoint-free.

Proof. Suppose $\{a, b, c\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$ satisfies $a < b < c$ and $a + c = 2b$. There is an integer $m \geq 0$ for which $3^m\{a, b, c\} \subset \mathbb{Z}_3^+(0, 1) + (1/2)$ so

$$2 \cdot 3^m\{a, b, c\} \subset 2\mathbb{Z}_3^+(0, 1) + 1. \quad (34)$$

This contradicts the fact that $2\mathbb{Z}_3^+(0, 1) + 1$ is midpoint-free, by Claim A. $\square$

Claim K. Any positive rational $b \in \mathbb{Q}_3^+([1]) \setminus \mathbb{Q}_3^+(0, 1)$ is the midpoint of a triple with endpoints in $\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$.

Proof. Given b , we seek $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$, such that $a < b < c$ and $a + c = 2b$. There is an integer m such that $3^m b \in \mathbb{Z}^+ + (1/2)$, so there is an integer $B \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0, 1)$ such that $3^m b = B + (1/2)$. At least one digit of B_3 is 2, so at least one digit of $(2B)_3$ is 1. Define $\{A, C\} \subset \mathbb{Z}_3^+(0, 1)$ by

$$\llbracket 2B \rrbracket_{3,i} = 0 \implies \llbracket A \rrbracket_{3,i} = \llbracket C \rrbracket_{3,i} = 0,$$

$$\llbracket 2B \rrbracket_{3,i} = 1 \implies \llbracket A \rrbracket_{3,i} = 0, \quad \llbracket C \rrbracket_{3,i} = 1, \quad (35)$$

$$\llbracket 2B \rrbracket_{3,i} = 2 \implies \llbracket A \rrbracket_{3,i} = \llbracket C \rrbracket_{3,i} = 1.$$

Then $\llbracket A \rrbracket_{3,i} \leq \llbracket C \rrbracket_{3,i}$ for all i , with strict inequality for at least one i , so $A < C$. Also $\llbracket A \rrbracket_{3,i} + \llbracket C \rrbracket_{3,i} = \llbracket 2B \rrbracket_{3,i}$ for all i , so $A + C = 2B$. Finally, let

$$3^m\{a, b, c\} = \{A, B, C\} + \frac{1}{2}. \quad (36)$$

Then $a + c = 2b$ and $\{a, c\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$. $\square$

Claim L. The set $(\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])) \cup -(\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1]))$ is midpoint-free.

Proof. By Claims G and J, if the specified set contains a midpoint triple, two members of the triple must have one sign and the third member must have the opposite sign. Suppose $\{a\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$ and $\{b, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ are such that $-a < b < c$ and $a + 2b = c$. Since a_3 , $(2b)_3$, and c_3 are eventually periodic, there is an integer m such that their digits in places $i \leq m$ are purely periodic, and

$$\begin{aligned} \llbracket a \rrbracket_{3,i} &= 1, & \llbracket 2b \rrbracket_{3,i} &\in \{0, 2\}, & \llbracket c \rrbracket_{3,i} &\in \{0, 1\} \\ & & & & \forall i \leq m. \end{aligned} \quad (37)$$

Since $c \notin \mathbb{Q}_3^+([1])$, there is a $j \leq m$ such that $\llbracket c \rrbracket_{3,j} = 0$ and therefore $\llbracket 2b \rrbracket_{3,j}$ cannot be equal to 0 for all $i \leq m$. On the other hand, $\llbracket 2b \rrbracket_{3,i}$ cannot be equal to 2 for all $i \leq m$ since $b \notin \mathbb{Q}_3^+([1])$. Hence the recurring block $[D]$ of $(2b)_3$ contains 0 and 2, so $\llbracket 2b \rrbracket_{3,k} = 0$, $\llbracket 2b \rrbracket_{3,k-1} = 2$ for some $k \leq m$. But $\llbracket a \rrbracket_{3,k} = \llbracket a \rrbracket_{3,k-1} = 1$, so $\llbracket a \rrbracket_{3,k-1} + \llbracket 2b \rrbracket_{3,k-1}$ has a “carry over” of 1, regardless of whether or not the digits in place $k-2$ contribute any “carry over”. Then

$$\llbracket c \rrbracket_{3,k} = \llbracket a \rrbracket_{3,k} + \llbracket 2b \rrbracket_{3,k} + (1) = 2 \notin \{0, 1\}. \quad (38)$$

By this contradiction, there is no midpoint triple of the proposed type. A similar but simpler argument shows that there are no subsets $\{a, b\} \subset \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])$ and $\{c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ such that $-a < -b < c$ and $a = 2b + c$. $\square$

Claim M. For any positive rational $a \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+([1])$ there is a midpoint triple in $(\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])) \cup \{-a\}$, with $-a$ as its lower endpoint.

Proof. Given a , we seek $\{b, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$ such that $-a < b < c$ and $a + 2b = c$. Note that if the triple $\{a, b, c\}$ has all the required properties, the scaled triple $3^m\{a, b, c\}$ also has all the required properties, for any integer m . Let us assume, without loss of generality, that the digits of a_3 in negative places are purely periodic, and partitioning into homogeneous blocks A_i yields

$$a_3 = A_h \cdots A_1 A_0 \cdot [A_{-1} A_{-2} \cdots A_{-k}] \quad (39)$$

for suitable integers $h \geq 0, k > 0$. As $a \notin \mathbb{Q}_3^+([1])$, if $k = 1$, then $A_{-1} = 0$ so $a \in \mathbb{Z}^+$ and a suitable set $\{b, c\}$ exists, by Claim D.

Now suppose $k \geq 2$ and a_3 has at least one negative place digit which is nonzero. If a_3 has no negative place digit equal to 2, let $a = A + q$, where $A \in \mathbb{Z}^+$ and $q \in \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$, $0 < q < 1$ are the integer part and fractional part of a , respectively. The proof of Claim D yields integers $B, C \in \mathbb{Z}_3^+(0, 1)$, such that $\{-A, B, C\}$ is a midpoint triple. Choose $b = B, c = C + q$. Then

$$\{-a, b, c\} = \{-A - q, B, C + q\} \quad (40)$$

is a midpoint triple with $\{b, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$.

Finally, suppose $A_{-j} = 2^{\times n}$ for at least one $j > 0$. Define homogeneous blocks B_i and C_i for all i as follows:

$$\begin{aligned} A_i = 0^{\times n} &\implies B_i = 1^{\times n}, & C_i = 0^{\times n} \\ A_i = 1^{\times n} &\implies B_i = 1^{\times n}, & C_i = 1^{\times n} \\ A_i = 2^{\times n} &\implies B_i = 0^{\times n}, & C_i = 0^{\times n}. \end{aligned} \quad (41)$$

Assuming a carry over digit equal to 1, note that checks as in the proof of Claim D justify the calculations

$$0^{\times n} + 2^{\times n} + (1) = (1) 0^{\times n}, \quad 1^{\times n} + 2^{\times n} + (1) = (1) 1^{\times n}, \quad (42)$$

so in all cases we have $A_i + 2B_i + (1) = (1)C_i$. Choose

$$\begin{aligned} b_3 &= B_h \cdots B_1 B_0 \cdot [B_{-1} B_{-2} \cdots B_{-k}], \\ c_3 &= (1) C_h \cdots C_1 C_0 \cdot [C_{-1} C_{-2} \cdots C_{-k}]. \end{aligned} \quad (43)$$

Then $a + 2b = c$ and $\{b, c\} \subset \mathbb{Q}_3^+(0, 1)$. Also $B_{-j} = C_{-j} = 0^{\times n}$ ensures that $\{b, c\} \subset \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$, as required. $\square$

For instance, beginning with

$$a = 2 \cdot [201]_3 = \frac{71}{26} \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+([1]), \quad (44)$$

the proof of Claim M constructs a midpoint triple $\{-a, b, c\}$ with

$$\begin{aligned} b &= 0 \cdot [011]_3 = \frac{2}{13} \in \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1]), \\ c &= 10 \cdot [001]_3 = \frac{79}{26} \in \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1]). \end{aligned} \quad (45)$$

However, if we begin with

$$a = 2 \cdot [101]_3 = \frac{31}{13} \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+([1]), \quad (46)$$

the construction falls back to Claim D, resulting in $b = 1$ and $c = 57/13$.

It now follows from Claims I, K, L, and M that the set

$$(\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])) \cup -(\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])) \quad (47)$$

is a maximal midpoint-free subset of $\mathbb{Q}$. This settles midpoint questions about $\mathbb{Q}_3^+(0, 1)$ as a subset of $\mathbb{Q}$.

What is the situation for $\mathbb{Q}_3^+(0, 2)$? If $q \in \mathbb{Q}_3^+(0, 2)$ then the recurring block of q_3 must have at least one digit equal to 0. Hence, there must be at least one digit equal to 0 in the recurring block of $(q/2)_3$, so $q/2 \in \mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$: it follows that $2(\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])) = \mathbb{Q}_3^+(0, 2)$. Again, if there is some $m \in \mathbb{Z}^+$, such that $3^m q \in \mathbb{Z}_3^+(0, 2) + 1$, it follows that q_3 is a terminating representation with trailing digit 1 and all other digits in $\{0, 2\}$. Hence,

$$\frac{q}{2} \in \mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1]). \quad (48)$$

The converse also holds, so

$$2(\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])) = \mathbb{Q}_3^+(0, 2; 1), \quad (49)$$

where $\mathbb{Q}_3^+(0, 2; 1)$ is the set of positive rationals which have a terminating base 3 representation with trailing digit 1 and all other digits in $\{0, 2\}$, so

$$\mathbb{Q}_3^+(0, 2; 1) = \bigcup_{i \in \mathbb{Z}^+} 3^{-i} (\mathbb{Z}_3^+(0, 2) + 1). \quad (50)$$

In passing, note that $\mathbb{Z}_3^+(0, 2) + 1 = \mathbb{Q}_3^+(0, 2; 1) \cap \mathbb{Z}^+$. By doubling, it now follows from Claim L that the set

$$\mathbb{Q}_3^+(0, 2) \cup -\mathbb{Q}_3^+(0, 2; 1) \quad (51)$$

is midpoint-free.

Claim N. Any rational $b \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2)$ is the midpoint of a triple with both its endpoints in $\mathbb{Q}_3^+(0, 2)$.

Proof. If $b \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2)$, then $b/2 \notin \mathbb{Q}_3^+(0, 1)$, so $b/2$ is the midpoint of a triple with endpoints in $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])$, by Claim I. By doubling, b is the midpoint of a triple with both its endpoints in the set $\mathbb{Q}_3^+(0, 2)$. $\square$

Claim O. For any rational $a \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2; 1)$, there is a midpoint triple in $\mathbb{Q}_3^+(0, 2) \cup \{-a\}$ with $-a$ as its lower endpoint.

Proof. If $a \in \mathbb{Q}^+ \setminus \mathbb{Q}_3^+(0, 2; 1)$ and $a > 0$, then $a/2 \notin \mathbb{Q}_3^+([1])$, so there is a midpoint triple in $\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1]) \cup \{-a/2\}$ with $-a/2$ as its lower endpoint, by Claim M. By doubling, $\mathbb{Q}_3^+(0, 2) \cup \{-a\}$ has a midpoint triple with lower endpoint $-a$. $\square$

The preceding results now fully establish the following.

Theorem 4. *The two sets*

$$\begin{aligned} &(\mathbb{Q}_3^+(0, 1) \setminus \mathbb{Q}_3^+([1])) \cup -(\mathbb{Q}_3^+(0, 1) \cap \mathbb{Q}_3^+([1])), \\ &\mathbb{Q}_3^+(0, 2) \cup -\mathbb{Q}_3^+(0, 2; 1) \end{aligned} \quad (52)$$

are both maximal midpoint-free subsets of $\mathbb{Q}$.

7. Midpoint-Free Subsets of $\mathbb{R}^+$ and $\mathbb{R}$

Now consider $\mathbb{R}_3^+(0, 1)$ and $\mathbb{R}_3^+(0, 2)$. These sets are uncountable, while their maximal rational subsets $\mathbb{Q}_3^+(0, 1)$ and $\mathbb{Q}_3^+(0, 2)$ are countable, yet no essentially new considerations arise in extending the results of the previous two sections from $\mathbb{Q}^+$ to $\mathbb{R}^+$ and from $\mathbb{Q}$ to $\mathbb{R}$. Noting that $\mathbb{R}_3^+([1]) = \mathbb{Q}_3^+([1])$ and $\mathbb{R}_3^+(0, 2; 1) = \mathbb{Q}_3^+(0, 2; 1)$, it suffices to state the end results for this wider context.

Theorem 5. *The sets $\mathbb{R}_3^+(0, 1) \setminus \mathbb{R}_3^+([1])$ and $\mathbb{R}_3^+(0, 2)$ are both maximal midpoint-free subsets of $\mathbb{R}^+$.*

Theorem 6. *The two sets*

$$\begin{aligned} &(\mathbb{R}_3^+(0, 1) \setminus \mathbb{R}_3^+([1])) \cup -(\mathbb{R}_3^+(0, 1) \cap \mathbb{R}_3^+([1])), \\ &\mathbb{R}_3^+(0, 2) \cup -\mathbb{R}_3^+(0, 2; 1) \end{aligned} \tag{53}$$

are both maximal midpoint-free subsets of $\mathbb{R}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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